Introduction to homological alg.

Ref. Ch. A. Weibel. An introduction to homological alg.

F. H. Croom. Basic concepts of algebraic topology

D. Eisenbud. Commutative algebra with a view toward algebraic geometry

T. W. Hungerford. Algebra

§ Localization of a commutative ring

<u>Def</u> R : comm. ring

(1) A subset $1 \in S$ of a ring R is multiplicative if $a, b \in S \Rightarrow a \cdot b \in S$.

(2) $(r_1, s_1) \sim (r_2, s_2)$ if $\exists s \in S$ s.t. $s(r_1 s_2 - r_2 s_1) = 0$. equiv. rel.

(3) $S^{-1}R := R \times S / \sim$

<u>Thm</u> (1) $S^{-1}R$ is a comm. ring with 1 & $\exists$ ring homo. $R \xrightarrow{\varphi} S^{-1}R$ & $\varphi(s)$ is a unit, $\forall s \in S$.

$$r \longmapsto \frac{r}{1}$$

(2) Let T be a comm. ring with 1. If $f: R \to T$ is a ring homo. s.t. $f(s)$ is a unit, $\forall s \in S$. Then $\exists!$ ring homo. $\bar{f}: S^{-1}R \to T$ s.t. $f = \bar{f} \circ \varphi$.

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr, "f"'] & S^{-1}R \arrow[d, dashed, "\exists!\bar{f}"] \\ & T \end{tikzcd}$$

§ Localization of a category

<u>Def</u> $\mathcal{C}$: cat. S : a collection of morphisms in $\mathcal{C}$.

A localization of $\mathcal{C}$ w.r.t. S is a cat. $S^{-1}\mathcal{C}$, together with a functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ s.t.

(1) $q(s)$ is an isom. in $S^{-1}\mathcal{C}$, $\forall s \in S$.

(2) Any functor $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(s)$ is an isom. $\forall s \in S$ factors thru q in a unique way.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "q"] \arrow[dr, "F"'] & S^{-1}\mathcal{C} \arrow[d, dashed, "\exists!"] \\ & \mathcal{D} \end{tikzcd}$$

$\Rightarrow$ $S^{-1}\mathcal{C}$ is unique up to equiv.

<u>Def</u> (multiplicative system) A collection S of morphisms in a cat $\mathcal{C}$ is called a multiplicative system in $\mathcal{C}$ if it satisfies the following 3 self-dual axioms:

1. S is closed under composition (if $s, t \in S$ are composable, then $st \in S$) & contains all identity morphisms ($\mathrm{id}_X \in S$, $\forall X \in \mathrm{Obj}\,\mathcal{C}$).

2. (Ore condition) If $t: Z \to Y$ is in S, then for every $g: X \to Y$ in $\mathcal{C}$, $\exists$ a comm. diag. "$gs = tf$" in $\mathcal{C}$ with $s \in S$.

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[d, "s"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[d, "t"] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}$$

(The slogan is "$t^{-1}g = fs^{-1}$ for some f & s".) Moreover the symm. statement (whose slogan is "$fs^{-1} = t^{-1}g$ for some t & g") is also valid.

3. (Cancellation) If $f, g : X \to Y$ are parallel morphisms in $\mathcal{C}$, then the following two conditions are equiv.

(a) $sf = sg$ for some $s \in S$ with source Y.

(b) $ft = gt$ for some $t \in S$ with target X.

Sketch of construction of $S^{-1}\mathcal{C}$.

there is no a priori reason for this to be a set

$\mathcal{C}, S$ as above $X, Y \in \mathrm{Ob}\,\mathcal{C}$

$$\rightsquigarrow \mathrm{Hom}_S(X,Y) = \left\{ X \xleftarrow{s} Z \xrightarrow{f} Y \right\} / \sim$$

equiv. rel.

$$\begin{tikzcd}[column sep=small] & Z_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ X & & Y \end{tikzcd} \quad \text{iff} \quad \exists \quad \begin{tikzcd} & Z_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ X & Z_3 \arrow[l] \arrow[u] \arrow[d] \arrow[r] & Y \\ & Z_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

$$\sim \begin{tikzcd}[column sep=small] & Z_2 \arrow[dl, "s_2"'] \arrow[dr, "f_2"] & \\ X & & Y \end{tikzcd}$$

Def A multiplicative system S is called locally small (on the left) if for each X, $\exists$ a set S_X of morphisms in S, all having target X, s.t. for every $X_1 \to X$ in S, $\exists$ a map $X_2 \to X_1$ in $\mathcal{C}$ s.t. the composite $X_2 \to X_1 \to X$ is in S_X.

Rmk If S is locally small, then $\mathrm{Hom}_S(X, Y)$ is a set. $\forall X, Y$.

Thm (Gabriel–Zisman thm) Let S be a locally small multiplicative system of morphisms in a cat. $\mathcal{C}$. Then the cat. $S^{-1}\mathcal{C}$ constructed above exists & is a localization of $\mathcal{C}$ w.r.t. S. The univ. functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ sends $f : X \to Y$ to $X \xleftarrow{id} X \xrightarrow{f} Y$.

Def (Quasi-isom.) A chain map $f : C_\cdot \to D_\cdot$ is a quasi-isom. if $H_n(f) : H_n(C_\cdot) \to H_n(D_\cdot)$ is an isom. $\forall n \in \mathbb{Z}$.

Rmk $f : C_\cdot \to D_\cdot$ qis. is not an isom. in $Ch(A)$.

Examples $f : C_\cdot \to D_\cdot$ is a qis.

$$
\begin{tikzcd}
C_\cdot : & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
D_\cdot : & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

$\mathbb{Z} \ni 1 \mapsto \bar{1} \in \mathbb{Z}/2$

However, $\mathrm{Hom}_{Ch(Ab)}(D_\cdot, C_\cdot) = 0$.

A : abelian cat. $\rightsquigarrow Ch(A)$

$\rightsquigarrow D(A) = Ch(A)[qis^{-1}]$.

§ The derived category

<u>Def</u> K: triangulated cat. $\mathcal{A}$: abelian cat.

An additive functor $H: K \to \mathcal{A}$ is called a covariant cohomological functor if whenever (u, v, w) is an ext. triangle on (A, B, C) the seq

$$\cdots \xrightarrow{w^*} H(T^i A) \xrightarrow{u^*} H(T^i B) \xrightarrow{v^*} H(T^i B) \longrightarrow H(T^{i+1} A) \xrightarrow{u^*} \cdots$$

is ext in $\mathcal{A}$. We often write $H^i(A)$ for $H(T^i A)$.

<u>Example</u> $H^0 : K(\mathcal{A}) \longrightarrow \mathcal{A}$

§ Construction of $D(A)$

K: triangulated cat. The system S arising from a coh. functor $H: K \to A$ is the collection of all morphisms s in K s.t. $H^i(s)$ is an isom. $\forall i$. (e.g. $S = \{qis\}$)

<u>Thm</u> If S arises from a cohomological functor, then

1. S is a multiplicative system.
2. $S^{-1}K$ is a triangulated cat & $K \to S^{-1}K$ is a morphism of triangulated cats (in any universe containing $S^{-1}K$).

$\mathcal{A} \rightsquigarrow Ch(\mathcal{A}) \rightsquigarrow K(\mathcal{A}) \rightsquigarrow D(\mathcal{A})$

Similarly $D^b(\mathcal{A})$, $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ are triangulated cats in any universe containing them.

§ Applications of derived categories

§ Derived functors

<u>Def</u> (Functor) $\mathcal{C}, \mathcal{D}$: categories.

A <u>functor</u> $F : \mathcal{C} \to \mathcal{D}$ is a rule that associates an obj. $F(C)$ of $\mathcal{D}$ to every obj C of $\mathcal{C}$ & a morphism $F(f) : F(C_1) \to F(C_2)$ in $\mathcal{D}$ to every morphism $f : C_1 \to C_2$ in $\mathcal{C}$. We require F to preserve identity morphisms ($F(id_C) = id_C$, $\forall C \in \mathrm{Obj}\,\mathcal{C}$) & composition ($F(g \circ f) = F(g) \circ F(f)$).

<u>Example</u> $Ch(\mathcal{A}) \xrightarrow{H_n(-)} \mathcal{A}$

Problem Most of the functors are not exact!

Examples

$$\begin{array}{c} \mathrm{Hom}_R(M,-) \\ R\text{-mods} \ni M \quad \xrightarrow{\hspace{3em}} \\ \mathrm{Hom}_R(-,M) \\ \xrightarrow{\hspace{3em}} \quad \mathrm{Ab} \\ \xrightarrow{\hspace{3em}} \\ -\otimes_R M \end{array}$$

are not exact in general.

Rmk (1) $\mathrm{Hom}_R(M,-)$ & $\mathrm{Hom}_R(-,M)$ are left ext.

(2) $-\otimes_R M$ is right ext.

Def (middle linear map) R: ring

A_R : right R-mod.

${}_R B$: left R-mod.

C : abelian gp.

A middle linear map from $A \times B$ to C is a ftn. $f : A \times B \longrightarrow C$ s.t.

$$\begin{cases} f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b) \\ f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2) \\ f(ar, b) = f(a, rb) \end{cases}$$

$\forall a, a_1, a_2 \in A, \forall b, b_1, b_2 \in B, \forall r \in R.$

Def (Tensor product) R: ring

A_R : right R-mod.

${}_RB$: left R-mod.

$$F = \sum \langle (a,b) \mid a \in A,\ b \in B \rangle$$

$$K = \left\langle \begin{array}{l} (a_1+a_2, b) - (a_1, b) - (a_2, b) \\ (a, b_1+b_2) - (a, b_1) - (a, b_2) \\ (ar, b) - (a, rb) \end{array} \middle| \begin{array}{l} a, a_1, a_2 \in A \\ b, b_1, b_2 \in B \\ r \in R \end{array} \right\rangle$$

$\leq F$

$$A \otimes_R B := F/K$$

<u>Thm</u> (1) $A \times B \xrightarrow{\varphi} A \otimes_R B$ is middle linear.

(2) If $f : A \times B \to C$ is a middle linear map,

then $\exists!\ \bar{f} : A \otimes_R B \to C$

s.t. $f = \bar{f} \circ \varphi$.

$$\begin{tikzcd} A \times B \arrow[r, "\varphi"] \arrow[dr, "f"'] & A \otimes_R B \arrow[d, dashed, "\exists!\ \bar{f}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & C \end{tikzcd}$$

Prop If $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an ext. seq. of left R-mods & D is a right R-mod, then $D\otimes_R A \xrightarrow{1\otimes f} D\otimes_R B \xrightarrow{1\otimes g} D\otimes_R C \to 0$ is ext.

Sketch of pf) (1) $1\otimes g$ is surj.

g is surj. $\Rightarrow \forall c\in C, \exists b\in B$ s.t. $g(b)=c$.

$\Rightarrow \underbrace{d\otimes c}_{\text{generator of } D\otimes_R C} \in \mathrm{Im}(1\otimes g) \Rightarrow 1\otimes g$ is surj. since it contains all generators.

(2) $g\circ f = 0 \Rightarrow (1\otimes g)\circ(1\otimes f) = 0$

$\Rightarrow \mathrm{Im}(1\otimes f) \subset \ker(1\otimes g)$.

(3) $D\otimes_R B/\mathrm{Im}(1\otimes f) \underset{\cong}{\overset{\exists!}{\rightleftarrows}} D\otimes_R B/\ker(1\otimes g) = D\otimes_R C$

$$\begin{tikzcd}
D\times C \arrow[rr] \arrow[dr] & & D\otimes_R C \arrow[dl, dashed, "\exists!"] \\
& D\otimes_R B/\mathrm{Im}(1\otimes f) &
\end{tikzcd}$$

① well-defined

② middle linear

$(d,c) \mapsto \overline{d\otimes b}$

$\Rightarrow \mathrm{Im}(1\otimes f) = \ker(1\otimes g)$ $\square$

Ex (1) $R=\mathbb{Z}$

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

$$\Big\Downarrow \ \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, -)$$

not surj!

$$0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z}$$

(2) $R=\mathbb{Z}$

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

$$\Big\Downarrow \ \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2\mathbb{Z})$$

not surj!

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \xrightarrow[\,=0\,]{\times 2} \mathbb{Z}/2\mathbb{Z}$$

(3) $R=\mathbb{Z}$

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

$$\Big\Downarrow \otimes_{\mathbb{Z}} \mathbb{Z}/2 \qquad \text{not injective!}$$

$$\mathbb{Z}/2 \xrightarrow[\,=\,0]{\times 2} \mathbb{Z}/2 \to \mathbb{Z}/2\mathbb{Z} \to 0$$

<u>Prop</u> $\mathcal{A} = R\text{-mod}$, $M : R\text{-mod}$.

$\mathrm{Hom}(M, -)$ & $\mathrm{Hom}(-, M)$ are left exact functors.

pf) (1) $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$: s.e.s.

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C)$$

$$\begin{array}{ccc} \mathrm{Hom}_R(M,A) & & \mathrm{Hom}_R(M,B) \\ \Psi & & \Psi \\ \varphi & \longmapsto & 0 \end{array}$$

M

φ $\quad$ $f \circ \varphi = 0$

$$0 \to A \xrightarrow{f} B \quad \Longrightarrow \quad \varphi = 0.$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(M,A) \arrow[r] & \mathrm{Hom}_R(M,B) \arrow[r] & \mathrm{Hom}_R(M,C) \\[-15pt]
& \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\[-15pt]
& \exists \psi \arrow[r, mapsto] & \varphi \arrow[r, mapsto] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
& & m \in M \arrow[dll, bend right, red] \arrow[dl, blue, "\exists\psi"'] \arrow[d, "\varphi"] \arrow[dr, "g\circ\varphi = 0"] & \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \\
& \exists!\, a \arrow[r, mapsto] & \varphi(m) \arrow[r, mapsto] & 0
\end{tikzcd}$$

□ (1)

(2) $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$: s.e.s.

$$\begin{tikzcd}
A \arrow[dr] & B \arrow[d] & C \arrow[dl] \\
& M &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(C,M) \arrow[r] & \mathrm{Hom}(B,M) \arrow[r] & \mathrm{Hom}(A,M) \\[-15pt]
& \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \\[-15pt]
& \varphi \arrow[r, mapsto] & g\circ\varphi &
\end{tikzcd}$$

$$0 = \varphi \circ g \qquad \begin{tikzcd} B \arrow[r, two heads, "g"] \arrow[d] & C \arrow[dl, "\varphi"] \\ M & \end{tikzcd} \qquad \begin{tikzcd} \exists b \arrow[r, mapsto] \arrow[d, mapsto] & c \arrow[dl, mapsto] \\ 0 & \end{tikzcd}$$

& g is surj.

$\Rightarrow \varphi = 0.$

$$0 \to \mathrm{Hom}(C, M) \to \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M)$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \varphi & \longmapsto & \varphi \circ f = 0 \end{array}$$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[dr] & B \arrow[r, "g"] \arrow[d, "\varphi"] & C \arrow[r] \arrow[dl] & 0 \\ & & M & & \end{tikzcd} \quad : \text{s.e.s.}$$

$$\begin{tikzcd} \exists b \arrow[r, mapsto] \arrow[d, mapsto] & c \arrow[dl, mapsto] \\ \varphi(b) & \end{tikzcd}$$

$\exists!\ \psi : C \to M$ s.t. $\varphi = \psi \circ g$.

$\Box$

Want Functors having better functorial behaviors.

{ Injective, projective objs

Def $\mathcal{A}$: abelian cat.

(1) An obj. P is projective if $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is ext.

(2) An obj I is injective if $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is injective.

Rmk An obj. P (resp. I) is projective (resp. inj.) iff it satisfies the following universal lifting property.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d] \\ B \arrow[r] & C \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl, dashed, "\exists"] \\ & I \end{tikzcd}$$

§ Injective, projective, free, flat mods

<u>Def</u> (1) A left R-mod M is <u>flat</u> if $- \otimes_R M$ is ext.

right N

$(N \otimes_R -)$

(2) A left R-mod F is <u>free</u> if it has a basis.

B : a set of basis.

R-mod

$$\begin{tikzcd} B \arrow[r] \arrow[dr] & F = R\langle B \rangle \arrow[d, dashed, "\exists!"] \\ & M \end{tikzcd}$$

<u>Prop</u> An R-mod is proj. iff it is a direct summand of a free R-mod.

pf) P : proj. & $F \twoheadrightarrow P$ $\rightsquigarrow$

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d, "\mathrm{id}"] & \\ F \arrow[r, two heads] & P \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow F = P \oplus Q$. □

Cor A free mod F is proj.

Baer's criterion. A right R-mod A is injective iff for every right ideal J of R, every map $J \to A$ can be extended to a map $R \to A$.

Cor $R = \mathbb{Z}$ or P.I.D.

An R-mod A is inj iff it is divisible, that is, for every $r \neq 0$ in R & every $a \in A$, $a = br$ for some $b \in A$.

Example $\mathbb{Q}$ & $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ are divisible in Ab. $\left(\mathbb{Z}[\frac{1}{p}] = \{ \frac{a}{p^n} \mid n \geq 1 \} \right)$

Thm (1) Projective mods are flat.

(2) Every finitely presented flat R-mod M is projective. (essential, e.g. $\mathbb{Q}$ is flat $\mathbb{Z}$-mod but not projective.)

Def $\mathcal{A}$: abelian cat.

(1) We say $\mathcal{A}$ has enough injectives if

$\forall A \in \mathrm{Obj}\,\mathcal{A}$, $\exists\ 0 \to A \to I$ with I : inj.

(2) We say $\mathcal{A}$ has enough projectives if

$\forall A \in \mathrm{Obj}\,\mathcal{A}$, $\exists\ P \to A \to 0$ with P : proj.

Example The cat. $\mathcal{A}$ of finite abelian gps is an abelian cat. with no projective objs.

Example R-mods is an abelian cat. with enough projectives & enough injectives.

Example Ab has enough injectives

A : abelian gp. $\rightsquigarrow I(A) = \prod_{\mathrm{Hom}_{\mathrm{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$

Check $I(A)$ is inj & $\exists\ A \hookrightarrow I(A)$.

Thm Suppose that $\mathcal{A}$ has enough injectives. Then $D^+(\mathcal{A})$ exists in our universe because it is equiv. to the full subcat $K^+(\mathcal{I})$ of $K^+(\mathcal{A})$ whose objs are bounded below cochain cpxes of injectives; $D^+(\mathcal{A}) \simeq K^+(\mathcal{I})$.

Dually, if $\mathcal{A}$ has enough projectives, then the localization $D^-(\mathcal{A})$ of $K^-(\mathcal{A})$ exists & is equiv. to the full subcat. $K^-(\mathcal{P})$ of bounded above cochain cpxes of projectives in $K^-(\mathcal{A})$: $D^-(\mathcal{A}) \cong K^-(\mathcal{P})$.

Motivation M : R-mod

free

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[rr] \arrow[dr, two heads] & & F_1 \arrow[rr, "\varphi_1"] \arrow[dr, two heads] & & F_0 \arrow[r, "\varphi_0"] & M \arrow[r] & 0 \\
& & \ker\varphi_1 \arrow[ur, hook] & & \ker\varphi_0 \arrow[ur, hook] & & &
\end{tikzcd}$$

§ Derived functor

<u>Def</u> (Derived functor)

$\mathcal{A}, \mathcal{B}$: abelian cat.

$\mathcal{A}$ has enough injectives (resp. projectives)

$F : \mathcal{A} \to \mathcal{B}$ left (resp. right) ext. functor.

$$\begin{tikzcd} D^+(\mathcal{A}) \simeq K^+(I) \arrow[r, "F"] & K^+(\mathcal{B}) \arrow[r] & D^+(\mathcal{B}) \end{tikzcd}$$

RF : the right derived functor.

$$\begin{tikzcd} D^-(\mathcal{A}) \simeq K^-(P) \arrow[r, "F"] & K^-(\mathcal{B}) \arrow[r] & D^-(\mathcal{B}) \end{tikzcd}$$

LF : left derived functor.

$$R^iF(A) = H^i(RF(A))$$
$$L^iF(A) = H^i(LF(A)).$$

§ Tor

Def Let B be a left R-mod. so that $T(A) = A \otimes_R B$ is a right ext. functor. $: \text{mod-}R \to Ab$.

$$\mathrm{Tor}_n^R(A, B) = (L_n T)(A).$$

Prop $\mathrm{Tor}_0(A, B) = A \otimes_R B$.

pf) Exercise.

§ Ext

<u>Def</u> For each R-mod A, the functor $F(B) = \mathrm{Hom}_R(A, B)$ is left ext. Its right derived functors are called the Ext gps &

$$\mathrm{Ext}^i_R(A,B) = R^i \mathrm{Hom}_R(A,-)(B).$$

$F : \mathcal{A} \longrightarrow \mathcal{B}$ contravariant left ext. functor

$\iff F : \mathcal{A}^{op} \longrightarrow \mathcal{B}$ covariant 〃 .

If $\mathcal{A}$ has enough projectives,
then $\mathcal{A}^{op}$ 〃 injectives.

$\rightsquigarrow$ We can define the right derived functor $R^iF(A)$ to be the coh. of $F(P.)$, $P. \longrightarrow A$ being a projective resol. in $\mathcal{A}$.

§ Ext & Extensions

Def (1) An ext. ξ of A by B is an ext. seq. $0 \to B \to X \to A \to 0$.

Two extensions ξ & ξ' are equiv. if $\exists$ a comm. diagram

$$\begin{tikzcd}
\xi : 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\cong"] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi' : 0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

(2) An extension is split if it is equiv. to $0 \to B \to A \oplus B \to A \to 0$.

$$b \mapsto (0,b)$$
$$(a,b) \mapsto a$$

Thm Given two R-mods A & B, the mapping $\Theta : \xi \mapsto \partial(\mathrm{id}_A)$ establishes a 1-1 Correspondence

$$\left\{\begin{array}{l}\text{equiv. classes of} \\ \text{exts of } A \text{ by } B\end{array}\right\} \xleftrightarrow{1-1} \mathrm{Ext}^1(A,B).$$

pf) Exercise. See Thm 3.4.3 of Weibel.

§ Mapping Cones & Cylinders

Def $f : B_\cdot \to C_\cdot$ a chain map.
The mapping cone of f is the chain cpx. $\mathrm{cone}(f)$ whose deg n part is $B_{n-1} \oplus C_n$.
The differential is given by the cpx.

$$\begin{bmatrix} -d_B & 0 \\ -f & d_C \end{bmatrix} : \begin{matrix} B_{n-1} \\ \oplus \\ C_n \end{matrix} \longrightarrow \begin{matrix} B_{n-2} \\ \oplus \\ C_{n-1} \end{matrix}$$

Check $\begin{bmatrix} -d_B & 0 \\ -f & d_C \end{bmatrix}\begin{bmatrix} -d_B & 0 \\ -f & d_C \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$B[p]_n = B_{n+p}$ with $(-1)^p d$

Rmk $\exists$ a s.e.s.

$$0 \to C \to \mathrm{Cone}(f) \xrightarrow{\delta} B[-1] \to 0$$

$$c \longmapsto (0, c)$$

$$(b, c) \longmapsto -b$$

$$0 \to C_{n+1} \to B_n \oplus C_{n+1} \to B_n \to 0$$

$$
\begin{tikzcd}
0 \arrow[r] & C_n \arrow[r] \arrow[d] & B_{n-1}\oplus C_n \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r] \arrow[d] & B_{n-2}\oplus C_{n-1} \arrow[r] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(b,c) \arrow[r, mapsto] \arrow[d, mapsto] & -b \arrow[d, mapsto] \\
 \arrow[r, mapsto] & db
\end{tikzcd}
$$

$$
\begin{bmatrix} -d_B & 0 \\ -f & d_C \end{bmatrix}\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} -db \\ dc - fb \end{bmatrix}
$$

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & B_n\oplus C_{n+1} \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n \arrow[r] \arrow[d] & B_{n-1}\oplus C_n \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r] \arrow[d] & B_{n-2}\oplus C_{n-1} \arrow[r] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$$
\begin{tikzcd}
c \arrow[r, mapsto] \arrow[d, mapsto] & (0,c) \arrow[d, mapsto] \\
dc \arrow[r, mapsto] & (0,dc)
\end{tikzcd}
$$

$\Downarrow$ homology l.e.s.

$$\cdots \to H_{n+1}(C_\cdot) \to H_{n+1}(\mathrm{Cone}(f)) \xrightarrow{\delta_*} H_{n+1}(B[-1]) \cong H_n(B)$$

$$\xrightarrow{\partial} H_n(C) \to H_n(\mathrm{Cone}(f)) \to H_{n-1}(B) \to \cdots$$

<u>Lemma</u> The map ∂ is f_*

pf) If $b \in B_n$ is a cycle, the elt. $(-b, 0) \in \mathrm{Cone}(f)_{n+1}$ lifts b via δ.

$$\Rightarrow \begin{bmatrix} -d_B & 0 \\ -f & d_C \end{bmatrix} \begin{bmatrix} -b \\ 0 \end{bmatrix} = \begin{bmatrix} db \\ fb \end{bmatrix} = \begin{bmatrix} 0 \\ fb \end{bmatrix}$$

$$\Rightarrow \partial[b] = [fb] = f_*[b].$$

$\square$

§ Cohomological functors

Def K: triangulated cat. $\mathcal{A}$: abelian cat.

An additive functor $H : K \to \mathcal{A}$ is called a covariant cohomological functor if whenever (u, v, w) is an ext. triangle on (A, B, C) the seq

$$\cdots \xrightarrow{w^*} H(T^i A) \xrightarrow{u^*} H(T^i B) \xrightarrow{v^*} H(T^i B) \longrightarrow H(T^{i+1} A) \xrightarrow{u^*} \cdots$$

is ext in $\mathcal{A}$. We often write $H^i(A)$ for $H(T^i A)$.

Example $H^0 : K(\mathcal{A}) \longrightarrow \mathcal{A}$

§ Examples

$$\mathrm{Ext}^1(\mathbb{Z}/2, \mathbb{Z}) \cong \mathbb{Z}/2 \cong \left\{ \begin{array}{l} 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0 \\[1em] 0 \to \mathbb{Z} \to \begin{array}{c}\mathbb{Z}\\ \oplus \\ \mathbb{Z}/2\end{array} \to \mathbb{Z}/2 \to 0 \end{array} \right.$$

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

qis.

$\mathrm{Hom}(-, \mathbb{Z})$

$$\begin{tikzcd}
\cdots & 0 \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d, equal] & \mathbb{Z} \arrow[l, "\times 2"'] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & 0 \arrow[l] & \mathbb{Z}/2 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

qis

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

$R\mathrm{Hom}(-, \mathbb{Z})$

$$\mathbb{Z}/2 \leftarrow \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \leftarrow \mathbb{Z}/2[-1]$$

$\mathrm{Hom}(-, \mathbb{Z})$

$$\mathbb{Z}/2 \leftarrow \mathbb{Z} \xleftarrow[\times 2]{} \mathbb{Z} \leftarrow 0$$

$$\cdots \leftarrow 0 \leftarrow 0 \leftarrow$$

§ Examples

$$\mathrm{Ext}^1(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2 \cong \left\{ \begin{array}{l} 0 \to \mathbb{Z}/2 \xrightarrow{\times 2} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0 \\ 0 \to \mathbb{Z}/2 \to \begin{array}{c}\mathbb{Z}/2 \\ \oplus \\ \mathbb{Z}/2\end{array} \to \mathbb{Z}/2 \to 0 \end{array} \right.$$

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

qis.

$\mathrm{Hom}(-, \mathbb{Z}/2)$

$$\begin{tikzcd}
\cdots & 0 \arrow[l] & \mathbb{Z}/2 \arrow[l] & \mathbb{Z}/2 \arrow[l, "\times 2 \,=\, 0"'] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

$\mathrm{Hom}(-, \mathbb{Z})$

$$\begin{tikzcd}
\cdots & 0 \arrow[l] & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \ .$$

$$\mathbb{Z}/2[-1] \to \mathbb{Z}/2[-1] \to \begin{array}{c}\mathbb{Z}/2[-1] \\ \oplus \\ \mathbb{Z}/2\end{array}$$

$\mathrm{RHom}(\mathbb{Z}/2, -)$

$\mathrm{Hom}(\mathbb{Z}/2, -)$

$$0 \to 0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \xrightarrow[0]{} \mathbb{Z}/2 \to \mathbb{Z}/2$$

§ Homological dimension

Def M : right R-mod.

(1) The projective dim $\mathrm{pd}(M)$ is the min. integer n (if it exists) s.t. $\exists$ a resol. of A by projective mods

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

(2) The injective dim $\mathrm{id}(M)$ is the min. integer n (if it exists) s.t. $\exists$ a resol. of A by projective mods

$$0 \to M \to I^0 \to I^1 \to \cdots \to I^n \to 0$$

(3) The flat dim $\mathrm{fd}(M)$ is the min. integer n (if it exists) s.t. $\exists$ a resol. of A by projective mods

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

<u>Thm</u> (Global dim thm) The following #s are the same for any ring R:

1. $\sup\{\mathrm{id}(B) \mid B \in \text{mod-}R\}$
2. $\sup\{\mathrm{pd}(A) \mid A \in \text{''}\ \}$
3. $\sup\{\mathrm{pd}(R/I) \mid I \subset R$ right ideal of $R\ \}$
4. $\sup\{d \mid \mathrm{Ext}_R^d(A,B) \neq 0$ for some right mods $A, B\}$

r. gl.dim(R)

This common # (possibly ∞) is called the (right) global dim of R. Bourbaki calls it the homological dim of R.

<u>Thm</u> (Tor-dim thm) The following #s are the same for any ring R:

1. $\sup\{\mathrm{fd}(B) \mid A \in \text{mod-}R\}$
2. $\sup\{\mathrm{fd}(R/I) \mid I \subset R\ \}$

2. $\sup\{fd(R/I) \mid I \subset R$ right ideal of R

3. $\sup\{fd(B) \mid A \in R\text{-mod}\}$

4. $\sup\{fd(R/I) \mid I \subset R$ left ideal of $R\}$

5. $\sup\{d \mid \mathrm{Tor}^d(A,B) \neq 0$ for some right mods $A, B\}$

This common # (possibly ∞) is called the (right) <u>Tor_dim of R.</u>

<u>Rmk</u> Projective mods are flat.

$\Longrightarrow$ $fd(A) \leq pd(A)$, $\forall A : R\text{-mod}.$

(not always $=$. For example $fd(\mathbb{Q}) = 0$ but $pd(\mathbb{Q}) = 1$)

$\Longrightarrow$ $\text{Tor\_dim}(R) \leq r.gl.dim(R)$

(not always $=$. When R is not noetherian)

<u>Def</u> A ring is right (left) noetherian if it satisfies the ascending chain condition on right (left) ideals.

Lemma TFAE for a right R-mod A

(1) $\mathrm{pd}(A) \le d$

(2) $\mathrm{Ext}_R^n(A,B)=0$ for all $n>d$ & all R-mods B.

(3) $\mathrm{Ext}_R^{d+1}(A,B)=0$ for all R-mods B.

(4) If $0 \to M_d \to P_{d-1} \to P_{d-2} \to \cdots \to P_1 \to P_0 \to A \to 0$ is any resol. with the P's projective, then M_d is also projective.

pf) It is clear that $(4) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3)$. If we are given a resol. of A as in (4), then $\mathrm{Ext}^{d+1}(A,B) \cong \mathrm{Ext}^1(M_d,B)$. M_d is proj. iff $\mathrm{Ext}^1(M_d,B)=0$, $\forall B$. $\therefore (3) \Rightarrow (4)$.

$\boxed{u}$

<u>Lemma</u> TFAE for a right R-mod A

(1) $fd(A) \leq d$

(2) $\mathrm{Tor}_n^R(A,B)=0$ for all $n>d$ & all R-mods B.

(3) $\mathrm{Tor}_{d+1}^R(A,B)=0$ for all R-mods B.

(4) If $0 \to M_d \to F_{d-1} \to F_{d-2} \to \cdots \to F_1 \to F_0 \to A \to 0$ is any resol. with the F's are flats, then M_d is also flat.

pf) It is clear that $(4) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3)$. If we are given a resol. of A as in (4), then $\mathrm{Tor}_{d+1}^R(A,B) \cong \mathrm{Tor}_1^R(M_d, B)$. M_d is flat iff $\mathrm{Tor}_1(M_d,B)=0$, $\forall B$. $\therefore$ $(3) \Rightarrow (4)$.

$\boxed{\text{Ц}}$

Prop TFAE for every left R-mod B.

1. B is a flat R-mod.
2. B^* is an inj. right R-mod.

The Pontrjagin dual B^* of a left R-mod B is the right R-mod. $\mathrm{Hom}_{Ab}(B, \mathbb{Q}/\mathbb{Z})$.

3. $I \otimes_R B \cong IB = \{x_1 b_1 + \cdots + x_n b_n \mid x_i \in I, b_i \in B\} \subset B$, $\forall I \subset R$ right ideal.
4. $\mathrm{Tor}_1^R(R/I, B) = 0$, $\forall I \subset R$ right ideal.

pf) $0 \to I \to R \to R/I \to 0$

$\Rightarrow 0 \to \mathrm{Tor}_1(R/I, R) \to I \otimes B \to B \to B/IB \to 0$

$\Rightarrow$ 3 is equiv. to 4.

For the other parts, see Prop 3.2.4 of Weibel.

Proof of Tor-dim thm)

Lemma $\Rightarrow$ $\sup(5) = \sup(1) \geq \sup(2)$.

The same lemma / R^{op} $\Rightarrow$ $\sup(5) = \sup(3)$ $\geq \sup(4)$. We may assume that $\sup(2) \leq \sup(4)$, that is, $d = \sup\{fd(R/J) \mid J$ is a right ideal$\} \leq \sup\{fd(R/J) \mid J$ is a ~~left~~ ideal$\}$.

$$\begin{cases} \text{If } d = \infty, & \checkmark \\ \text{If } d < \infty. & \end{cases}$$

Suppose that $fd(B) > d$. For this B, choose a resol. $0 \to M \to F_{d-1} \to \cdots \to F_0 \to B \to 0$ with F's flat. But then for all ideals J we have

$$0 = \mathrm{Tor}^R_{d+1}(R/J, B) = \mathrm{Tor}^R_1(R/J, M).$$

$\Rightarrow$ M is flat. $\Rightarrow$ $fd(B) \leq d$. $\divideontimes$

$\boxed{4}$

Prop If R is right noetherian, then

(1) $\mathrm{fd}(A) = \mathrm{pd}(A)$, $\forall A$: f.g. R-mod.

(2) $\mathrm{Tor\text{-}dim}(R) = \mathrm{r.gl.dim}(R)$.

pf) Since we can compute $\mathrm{Tor\text{-}dim}(R)$ & $\mathrm{r.gl.dim}(R)$ using the modules R/I, it suffices to prove (1). Since $\mathrm{fd}(A) \leq \mathrm{pd}(A)$, it suffices to prove $\mathrm{pd}(A) \leq n$ if $\mathrm{fd}(A) \leq n$.

As R is noetherian, $\exists$ a resol.

$$0 \to M \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

in which P_i are f.g. free mods & M is finitely presented.

$\underset{\text{fd Lemma 4.1.10}}{\Longrightarrow}$ M is a flat R-mod

$\Rightarrow$ M is projective. $\Rightarrow$ $\mathrm{pd}(A) \leq n$.

$\boxed{}$

Def (1) A ring R is quasi-Frobenius if it is (left & right) noetherian & R is an injective (left & right) R-mod.

(2) A Frobenius alg / k: field is a finite dim'l alg R s.t. $R \cong \mathrm{Hom}_k(R, k)$ as right R-mods.

Thm (Faith, Faith-Walker) TFAE for every ring R:

1. R is quasi-Frobenius
2. Every proj. right R-mod is inj.
3. " left " .
4. Every inj. right " proj.
5. " left " .

§ Dimension of a comm. ring

R: comm. noetherian ring

Def R is a local ring if $\exists!$ max. ideal $\mathfrak{m}$ & $k = R/\mathfrak{m}$.

Def (1) The Krull dim of a ring R, $\dim R$ is the length of the longest chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d$ of prime ideals in R.

(2) The embedding dim. of a local ring R is $\text{emb.dim}(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$

(3) A local ring is called a regular local ring if $\dim R = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. (In general $\dim R \leq \text{emb.}\dim_k(\mathfrak{m}/\mathfrak{m}^2)$)

(4) M: f.g. R-mod. A regular seq. on M or M-seq. is a seq. $(x_1, \cdots, x_n)$

in M s.t. x_1 is a nonzerodiv. in M & x_i is a nonzerodiv. in $M/(x_1, \cdots, x_{i-1})M$ for $i > 1$.

(5) The depth of M, $\operatorname{depth}(M)$ is the length of the longest regular seq. on M. (For any local ring R, we have $\operatorname{depth}(R) \leq \dim R$.)

(6) R is called Cohen-Macaulay if $\operatorname{depth}(R) = \dim R$.

Thm A local ring is regular iff gl. $\dim(R) < \infty$. In this case

$$\operatorname{depth}(R) = \dim(R) = \operatorname{emb.dim}(R) = \operatorname{gl.dim}(R)$$
$$= \operatorname{pd}_R(k).$$

§ Group homology & cohomology

Def G : gp, R : comm. ring

A gp ring $R[G] = \{ \sum_{\text{finite}} r_i g_i \}$

Def G : gp. A (left) G-mod is an abelian gp. A on which G acts by additive maps on the left.

$$G \times A \longrightarrow A$$
$$(g, a) \longmapsto ga$$

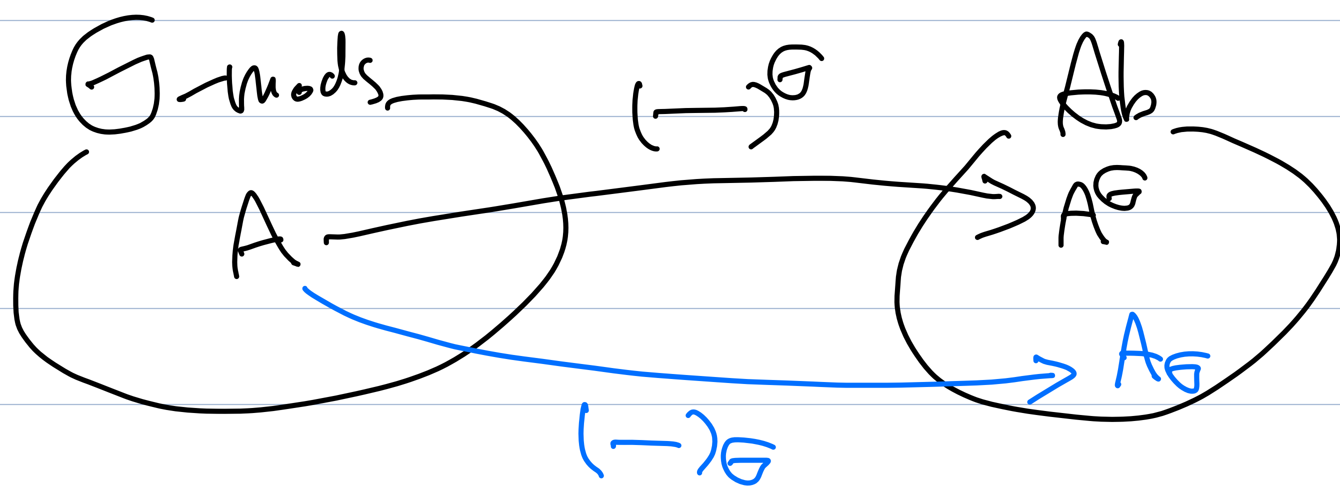
Rmk G-mod $\simeq \mathbb{Z}G$-mod.

§ Examples

Def (1) A trivial G-mod is an abelian gp A on which G acts trivially, that is $ga = a$, $\forall g \in G$, $\forall a \in A$.

(2) The inv. subgp A^G of a G-mod A is
$$A^G = \{ a \in A \mid ga = a, \ \forall g \in G \}$$

(3) The coinvariants A_G of a G-mod A is
$$A_G = A / \text{submod gen. by } \{ ga - a, \ g \in G, \ a \in A \}$$



§ Group homology & cohomology

Lemma A : any G-mod.

$\mathbb{Z}$: trivial G-mod.

$\Rightarrow$ ① $A_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} A$

& ② $A^G \cong \operatorname{Hom}_G(\mathbb{Z}, A)$.

pf) ① Considering $\mathbb{Z}$ as $\mathbb{Z}-\mathbb{Z}G$ bimod, the "trivial mod functor" : $\mathbb{Z}\text{-mod} \longrightarrow \mathbb{Z}G\text{-mod}$ is the functor $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, -)$. Its left adjoint is $\mathbb{Z} \otimes_{\mathbb{Z}G} -$ & $(-)_G$.

$\Rightarrow \mathbb{Z} \otimes_{\mathbb{Z}G} - \cong (-)_G$.

adjoint

② $A^G \cong \operatorname{Hom}_{Ab}(\mathbb{Z}, A^G) \cong \operatorname{Hom}_G(\mathbb{Z}, A)$

$\Rightarrow \operatorname{Hom}_G(\mathbb{Z}, -) \cong (-)^G$.

$\square$

Def A: G-mod. We write $H_*(G;A)$ for the left derived functors $L_*(-_G)(A)$ & called them homology gps of G with coefficients in A; by the lemma above, $H_*(G,A) \cong \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z},A)$.
By definition, $H_0(G,A) = A_G$.

Similarly, we write $H^*(G,A)$ for the right derived functor $R^*(-^G)(A)$ & call them the coh. gps of G with coefficients in A; by lemma above $H^*(G,A) \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z},A)$. By definition $H^0(G,A) = A^G$.

§ Cyclic & free gps

$$C_m := \langle \sigma \rangle \cong \mathbb{Z}/m\mathbb{Z}$$

$$N := 1 + \sigma + \sigma^2 + \cdots + \sigma^{m-1}$$

$$\Rightarrow (\sigma - 1)N = \sigma^m - 1 = 0 \in \mathbb{Z}C_m.$$

$\mathbb{Z}$: trivial C_m-mod.

$$\cdots \xrightarrow{N} \mathbb{Z}C_m \xrightarrow{\sigma-1} \mathbb{Z}C_m \xrightarrow{N} \mathbb{Z}C_m \xrightarrow{\sigma-1} \mathbb{Z}C_m \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$\Downarrow$ periodic free resol.

Thm $G = C_m$ & A: G-mod.

$$H_n(G,A) = \begin{cases} A/(\sigma-1)A & \text{if } n=0 \\ A^G/NA & \text{if } n=1,3,5,\cdots \\ \{a \in A \mid Na = 0\}/(\sigma-1)A & \text{if } n=2,4,6,\cdots \end{cases}$$

&

$$H^n(G,A) = \begin{cases} A^G & \text{if } n=0 \\ \{a \in A \mid Na = 0\}/(\sigma-1)A & \text{if } n=1,3,5,\cdots \\ A^G/NA & \text{if } n=2,4,6,\cdots \end{cases}$$

§ Applications

<u>Def</u> (Classifying sp.) A CW cpx with fundamental gp. G & contractible univ. covering sp. is called a classifying sp for G, or a model for BG; by abuse of notation, we will call such a sp. BG & write EG for its univ. covering sp.

$$\underset{\text{Serre fib.}}{\Longrightarrow} \quad \pi_i(BG) = \begin{cases} G & \text{if } i=1 \\ 0 & \text{o.w} \end{cases}$$

<u>Fact</u> Any two classifying sps for G are htpy equiv. (e.g. $B\mathbb{Z} = S^1 \underset{\text{htpy}}{\simeq} \mathbb{C}^\times$)

<u>Thm</u> $H_*(BG, \mathbb{Z}) \cong H_*(G, \mathbb{Z})$

& $H^*(BG, \mathbb{Z}) \cong H^*(G, \mathbb{Z})$

pf) Since $H_*(EG) \cong H_*(pt)$ is 0 for $* \neq 0$ & $\mathbb{Z}$ for $*=0$, the chain cpx $S_*(EG)$ is a free $\mathbb{Z}G$-mod resol. of $\mathbb{Z}$.

$\Rightarrow H_*(G,\mathbb{Z}) = H_*(S_*(EG) \otimes_{\mathbb{Z}G} \mathbb{Z})$

$= H_*(S_*(EG)_G) = H_*(S_*(BG))$

$= H_*(BG, \mathbb{Z})$.

Similarly, $H^*(G,\mathbb{Z})$ is the coh. of

$\mathrm{Hom}_G(S_*(EG), \mathbb{Z}) = \mathrm{Hom}_{Ab}(S_*(EG)_G, \mathbb{Z})$

$= \mathrm{Hom}_{Ab}(S_*(BG), \mathbb{Z})$ the chain cpx.

whose coh. is $H^*(BG, \mathbb{Z})$. $\boxed{\text{L}}$

§ Equivariant homology & cohomology

X : top'l sp. $\circlearrowleft G$

$$H^*_G(X,\mathbb{Z}) := H^*(X \times_G EG, \mathbb{Z})$$

$$\begin{tikzcd} X \times EG \arrow[r] \arrow[d] & EG \arrow[d] \\ X \times_G EG \arrow[r] & BG \end{tikzcd}$$

$$\Longrightarrow \quad H^*(BG,\mathbb{Z}) \longrightarrow H^*_G(X,\mathbb{Z})$$

<u>Thm</u> Let G acts on a sp. X with $\pi_0(X)=0$.

Then for every abelian gp. A there are s.s.

$$\mathrm{I}\,E^2_{p,q} = H_p(G, H_q(X,A)) \Longrightarrow H_{p+q}(X/G, A);$$

$$\mathrm{II}\,E^2_{p,q} = H^p(G, H^q(X,A)) \Longrightarrow H^{p+q}(X/G, A).$$